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The Pinsker subgroup of an algebraic flow

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ABSTRACT

The algebraic entropy h defined for endomorphisms ϕ of abelian groups G measures the growth of the trajectories of non-empty finite subsets F of G with respect to ϕ. We show that this growth can be either polynomial or exponential. The greatest ϕ-invariant subgroup of G where this growth is polynomial coincides with the greatest ϕ-invariant subgroup $\mathbf{P}(G, \phi)$ of G (named Pinsker subgroup of ϕ) such that $h(\phi \restriction_{\mathbf{P}(G,\phi)}) = 0$. We obtain also an alternative characterization of $\mathbf{P}(G, \phi)$ from the point of view of the quasi-periodic points of ϕ.

1. Introduction

The algebraic entropy ent of endomorphisms of abelian groups was first defined by Adler et al. in [1], then studied by Weiss in [23] and more recently in [6]. This function is studied also in other recent papers ([3,11,18,19,26]).

The definition given by Weiss is appropriate for *torsion* abelian groups, since $\mathrm{ent}(\phi) = \mathrm{ent}(\phi \restriction_{t(G)})$ for any endomorphism ϕ of an abelian group G (so endomorphisms of torsion-free abelian groups have zero algebraic entropy). Peters [16] modified the definition of algebraic entropy for automorphisms of arbitrary abelian groups, using the non-empty finite subsets instead of the finite subgroups used by Weiss. This definition can be extended to endomorphisms of abelian groups, as follows. Let G be an abelian group and $\phi \in \mathrm{End}(G)$. For a non-empty subset F of G and for any positive integer n, the nth *ϕ-trajectory* of F is

$$T_n(\phi, F) = F + \phi(F) + \cdots + \phi^{n-1}(F),$$

and the *ϕ-trajectory* of F is $T(\phi, F) = \sum_{n\in\mathbb{N}} \phi^n(F)$. For F finite, let

$$\tau_{\phi,F}(n) = |T_n(\phi, F)|;$$

when there is no possibility of confusion we write only $\tau_F(n)$, omitting the endomorphism ϕ. For the function $\tau_{\phi,F} : \mathbb{N}_+ \to \mathbb{N}$ the limit

$$H(\phi, F) = \lim_{n\to\infty} \frac{\log \tau_{\phi,F}(n)}{n} \tag{1.1}$$

exists (see Claim 2.1(a) or [5]), and $H(\phi, F)$ is the *algebraic entropy of ϕ with respect to F*. The *algebraic entropy* of ϕ is

$$h(\phi) = \sup\{H(\phi, F) : F \subseteq G \text{ non-empty finite}\}.$$

When ϕ is an automorphism, the algebraic entropy defined in this way has the same values as that introduced (in a different way) by Peters. Clearly, since ent is defined by $\mathrm{ent}(\phi) = \sup\{H(\phi, F) : F \subseteq G \text{ finite subgroup}\}$, h coincides with ent for

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endomorphisms of torsion abelian groups, but while the function ent trivializes for torsion-free abelian groups, h remains non-trivial also in this case. Many properties satisfied by ent are shared also by h (see Fact 2.5).

For a measure preserving transformation ϕ of a measure space $(X, \mathcal{B}, \mu)$ the *Pinsker σ-algebra* $\mathfrak{P}(\phi)$ of ϕ is the greatest σ-subalgebra of $\mathcal{B}$ such that ϕ restricted to $(X, \mathfrak{P}(\phi), \mu \restriction_{\mathcal{B}})$ has zero measure entropy. Note that $id_X : (X, \mathfrak{B}, \mu) \to (X, \mathfrak{P}(\phi), \mu \restriction_{\mathcal{B}})$ is measure preserving, so $(X, \mathfrak{P}(\phi), \mu \restriction_{\mathcal{B}})$ is a factor of $(X, \mathfrak{B}, \mu)$ (see [21]).

The situation is similar in topological dynamics. Let (X, ϕ) be a *topological flow*, i.e., a homeomorphism $\phi : X \to X$ of a compact Hausdorff space X. A *factor* $(\pi, (Y, \psi))$ of (X, ϕ) is a topological flow (Y, ψ) together with a continuous surjective map $\pi : X \to Y$ such that $\pi \circ \phi = \psi \circ \pi$. In [4] (see also [13]) it is proved that a topological flow (X, ϕ) admits a largest factor with zero topological entropy, called *topological Pinsker factor*.

In analogy with the topological case, we call *algebraic flow* a pair (G, ϕ), where G is an abelian group and $\phi \in \mathrm{End}(G)$. For an algebraic flow (G, ϕ), we say that a subgroup H of G is *ϕ-invariant* (or just *invariant* when ϕ is clear from the context) if $\phi(H) \subseteq H$. A *factor* of (G, ϕ) is an algebraic flow of the form $(G/H, \overline{\phi})$, where H is a ϕ-invariant subgroup of G and $\overline{\phi}$ is the endomorphism induced by ϕ on the quotient G/H.

An algebraic flow may fail to have a largest factor with zero algebraic entropy, even when the underlying group is torsion, as Example 3.5 shows. So the algebraic counterpart of the Pinsker factor in the case of an algebraic flow cannot be a *factor*. This motivated us to introduce the notion of Pinsker subgroup as follows.

Definition 1.1. Let (G, ϕ) be an algebraic flow. The *Pinsker subgroup* of G with respect to ϕ is the greatest ϕ-invariant subgroup $\mathbf{P}(G, \phi)$ of G such that $h(\phi \restriction_{\mathbf{P}(G,\phi)}) = 0$.

As shown by Proposition 3.1, the Pinsker subgroup exists and has a number of nice properties. The aim of this paper is to study the properties of this subgroup and characterize it in two distinct ways.

The first one involves the very definition of algebraic entropy measuring the growth of the function $\tau_{\phi,F}(n)$. Since

$$|F| \leq \tau_{\phi,F}(n) \leq |F|^n \text{ for every } n \in \mathbb{N}_+, \tag{1.2}$$

the growth of $\tau_{\phi,F}$ is always at most exponential; moreover, $H(\phi, F) \leq \log |F|$ in (1.1) (note that (1.2) implies also that $\tau_{\phi,F}(n) = 1$ for every $n \in \mathbb{N}_+$ precisely when F is a singleton). This justifies the following definition.

Definition 1.2. Let (G, ϕ) be an algebraic flow and let F be a non-empty finite subset of G. Then we say that:

(a) ϕ has *exponential growth* with respect to F (denoted by $\phi \in \mathrm{Exp}_F$) if there exists $b \in \mathbb{R}$, $b > 1$, such that $\tau_{\phi,F}(n) \geq b^n$ for every $n \in \mathbb{N}_+$;
(b) ϕ has *polynomial growth* with respect to F (denoted by $\phi \in \mathrm{Pol}_F$) if there exists $P_F(X) \in \mathbb{Z}[X]$ such that $\tau_{\phi,F}(n) \leq P_F(n)$ for every $n \in \mathbb{N}_+$;
(c) ϕ has *polynomial growth* (denoted by $\phi \in \mathrm{Pol}$) if $\phi \in \mathrm{Pol}_F$ for every non-empty finite subset F of G.

It is not hard to see that for an algebraic flow (G, ϕ) and a fixed non-empty finite subset F, ϕ has exponential growth with respect to F if and only if $H(\phi, F) > 0$ (see Theorem 6.12). On the other hand, if $H(\phi, F) = 0$, then $\tau_{\phi,F}$ has growth less than exponential, yet it is not clear whether $\phi \in \mathrm{Pol}_F$ holds. To clarify this issue, in Section 5 we show that for any algebraic flow (G, ϕ) there exists a greatest ϕ-invariant subgroup $\mathrm{Pol}(G, \phi)$ of G where the restriction of ϕ has polynomial growth. Since $\phi \in \mathrm{Pol}_F$ immediately yields $H(\phi, F) = 0$ (see Corollary 6.2), this entails $\mathrm{Pol}(G, \phi) \subseteq \mathbf{P}(G, \phi)$.

We show that these two subgroups coincide as a corollary of our Main Theorem (see below) involving also another important subgroup related to the algebraic flow (G, ϕ). This characterizes the Pinsker subgroup as the greatest ϕ-invariant subgroup of G on which the restriction of ϕ has polynomial growth. Moreover, using the equality $\mathrm{Pol}(G, \phi) = \mathbf{P}(G, \phi)$ we deduce the following surprising

Dichotomy Theorem. *Every algebraic flow (G, ϕ) has either exponential or polynomial growth with respect to any fixed non-empty finite subset F of G.*

In other words, $H(\phi, F) = 0$ if and only if $\phi \in \mathrm{Pol}_F$ holds, for any fixed non-empty finite subset F of G. The Dichotomy Theorem is stronger than the equality $\mathbf{P}(G, \phi) = \mathrm{Pol}(G, \phi)$, since by taking the universal quantifiers with respect to F, one can easily deduce that $h(\phi) = 0$ if and only if $\phi \in \mathrm{Pol}$ (see Corollary 6.11), i.e., $\mathbf{P}(G, \phi) = \mathrm{Pol}(G, \phi)$. The proof of the Dichotomy Theorem requires a significant effort (see Theorem 6.12).

The function $\tau_{\phi,F}$ makes sense for an endomorphism ϕ of a non-abelian group G and a non-empty finite subset F of G. Nevertheless, here the Dichotomy Theorem may fail even for the identity endomorphism of G (see Remark 2.3).

The second way to characterize the Pinsker subgroup involves another (more immediate) aspect of the dynamics of an algebraic flow (G, ϕ), namely, the quasi-periodic points of ϕ in G (an element $x \in G$ is a *quasi-periodic* point of ϕ if there exist $n > m$ in $\mathbb{N}$ such that $\phi^n(x) = \phi^m(x)$). For an algebraic flow (G, ϕ) the *ϕ-torsion subgroup* of G

$$t_\phi(G) = \{x \in G : |T(\phi, \langle x \rangle)| < \infty\}$$

was introduced in [6]. Note that $t_\phi(G) \subseteq t(G)$, and $t_\phi(G)$ consists of the quasi-periodic points of ϕ in $t(G)$. The equality $\mathbf{P}(G, \phi) = t_\phi(G)$ for a torsion abelian group G can be attributed to [6] (see also Corollary 3.6 for a proof). In other words, in this case the Pinsker subgroup coincides with the subgroup of all quasi-periodic points.

In Section 4 we elaborate this key idea in the general case, by introducing the counterpart of $t_\phi(G)$ in the non-torsion case. Namely, we introduce the smallest ϕ-invariant subgroup $\mathfrak{Q}(G,\phi)$ of G such that the induced endomorphism $\overline{\phi}$ of $G/\mathfrak{Q}(G,\phi)$ has no non-zero quasi-periodic points (so in particular, it contains all quasi-periodic points of ϕ in G). This subgroup captures important features of the dynamics of the endomorphism ϕ from this internal point of view, making no recurse to the algebraic entropy of ϕ. In Section 5 we show that $\mathfrak{Q}(G,\phi)\subseteq \mathrm{Pol}(G,\phi)$ (see Corollary 5.11). Along with the above mentioned inclusion $\mathrm{Pol}(G,\phi)\subseteq \mathbf{P}(G,\phi)$ this gives a chain $\mathfrak{Q}(G,\phi)\subseteq \mathrm{Pol}(G,\phi)\subseteq \mathbf{P}(G,\phi)$. In the key Theorem 6.6 we prove that if $\mathbf{P}(G,\phi)=G$, then $\mathfrak{Q}(G,\phi)\neq 0$. From this it is possible to deduce that all three subgroups coincide:

Main Theorem. *For every algebraic flow (G,ϕ),*

$$\mathfrak{Q}(G,\phi)=\mathrm{Pol}(G,\phi)=\mathbf{P}(G,\phi).$$

This alternative description of the Pinsker subgroup is proved in Theorem 6.9.

Notation and terminology

We denote by $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_+$, $\mathbb{Q}$ and $\mathbb{R}$ respectively the set of integers, the set of natural numbers, the set of positive integers, the set of rationals and the set of reals. For $m\in\mathbb{N}_+$, we use $\mathbb{Z}(m)$ for the finite cyclic group of order m.

Let G be an abelian group. With a slight divergence from the standard use, we denote by $[G]^{<\omega}$ the set of all non-empty finite subsets of G. If H is a subgroup of G, we indicate this by $H\leq G$. The subgroup of torsion elements of G is $t(G)$, while $D(G)$ denotes the divisible hull of G. For a cardinal α we denote by $G^{(\alpha)}$ the direct sum of α many copies of G, that is, $\bigoplus_\alpha G$.

Moreover, $\mathrm{End}(G)$ is the ring of all endomorphisms of G. We denote by 0_G and id_G respectively the endomorphism of G which is identically 0 and the identity endomorphism of G. For $F\in[G]^{<\omega}$ and $n\in\mathbb{N}_+$ let

$$F_{(n)}=\underbrace{F+\cdots+F}_{n}.$$

The hyperkernel of $\phi\in\mathrm{End}(G)$ is $\ker_\infty\phi=\bigcup_{n\in\mathbb{N}}\ker\phi^n$. The endomorphism $\overline{\phi}:G/\ker_\infty\phi\to G/\ker_\infty\phi$ induced by ϕ is injective.

2. Background on algebraic entropy

For an algebraic flow (G,ϕ) and $F\in[G]^{<\omega}$, since $T(\phi,F)$ need not be a subgroup of G, we denote by $V(\phi,F)=\langle T(\phi,F)\rangle=\langle\phi^n(F):n\in\mathbb{N}\rangle$ the smallest ϕ-invariant subgroup of G containing F (and consequently containing also $T(\phi,F)$). If $F=\{g\}$ for some $g\in G$ we write simply $V(\phi,g)$. Note that $V(\phi,F)=\sum_{g\in F}V(\phi,g)$.

It is clear from the definition that, for the computation of the algebraic entropy of an endomorphism ϕ of an abelian group G, it is not restrictive to consider subsets $F\in[G]^{<\omega}$ such that $0\in F$ and $F=-F$.

In the next claim we collect some technical properties of the trajectories that will be used in the paper.

Claim 2.1. *Let (G,ϕ) be an algebraic flow, $F\in[G]^{<\omega}$ such that $0\in F$, and let $n,k\in\mathbb{N}_+$. Then:*

(a) *$T_{n+k}(\phi,F)=T_n(\phi,F)+\phi^n(T_k(\phi,F))$ (consequently, $\tau_{\phi,F}(n+k)\leq\tau_{\phi,F}(n)\tau_{\phi,F}(k)$; in particular, the limit in (1.1) exists);*
(b) *$T_n(\phi^k,F)\subseteq T_{nk-k+1}(\phi,F)$ (consequently, $\tau_{\phi^k,F}(n)\leq\tau_{\phi,F}(nk-k+1)$);*
(c) *$T_{nk}(\phi,F)=T_n(\phi^k,T_k(\phi,F))$ (consequently, $\tau_{\phi,F}(nk)=\tau_{\phi^k,T_k(\phi,F)}(n)$).*
(d) *In particular, $T(\phi,F)=T(\phi^k,T_k(\phi,F))$.*
(e) *Moreover, for every $F_1,F_2\in[G]^{<\omega}$, $T_n(\phi,F_1\cup F_2)\subseteq T_n(\phi,F_1)+T_n(\phi,F_2)$, and so $H(\phi,F_1\cup F_2)\leq H(\phi,F_1)+H(\phi,F_2)$. In particular, $H(\phi,F\cup -F)\leq 2H(\phi,F)$.*

Proof. (a) The equality and the inequality are obvious. They show that the sequence $\{c_n:n\in\mathbb{N}_+\}$, where $c_n=\log\tau_{\phi,F}(n)$, satisfies $c_{n+k}\leq c_n+c_k$ for $n,k\in\mathbb{N}_+$, and so Fekete's Lemma [10] applies to conclude that the limit $H(\phi,F)=\lim_{n\to\infty}\frac{c_n}{n}$ exists (and coincides with $\inf_{n\in\mathbb{N}_+}\frac{c_n}{n}$).

The remaining items (b), (c) and (d) are easy to check. For (e) note that $H(\phi,F)=H(\phi,-F)$. □

In the next claim we see that the identity endomorphism of any abelian group has polynomial growth and we deduce in Example 2.4 that the algebraic entropy of the identity endomorphism is zero.

Claim 2.2. *Let G be an abelian group and $F\in[G]^{<\omega}$. Then $|F_{(n)}|\leq(n+1)^{|F|}$ for every $n\in\mathbb{N}_+$.*

Proof. Let $F=\{f_1,\dots,f_t\}$ and let $n\in\mathbb{N}_+$. If $x\in F_{(n)}$, then $x=\sum_{i=1}^t m_if_i$, for some $m_i\in\mathbb{N}$ with $\sum_{i=1}^t m_i=n$. Then $0\leq m_i\leq n$ for every $i=1,\dots,t$, that is, $(m_1,\dots,m_t)\in\{0,1,\dots,n\}^t$, and so $|F_{(n)}|\leq(n+1)^t$. □

Remark 2.3. The growth of the function $n\mapsto|F_{(n)}|$ has been extensively studied for non-abelian groups G, where it may fail to be polynomial [24].

Example 2.4. For any abelian group G, the identity endomorphism id_G has $h(id_G) = 0$. Indeed, let $F \in [G]^{<\omega}$. By Claim 2.2, $\tau_{id_G,F}(n) = |T_n(id_G, F)| = |F_{(n)}| \leq (n+1)^{|F|}$ for every $n \in \mathbb{N}_+$. Hence,

$$H(id_G, F) = \lim_{n\to\infty} \frac{\log \tau_{id_G,F}(n)}{n} \leq \lim_{n\to\infty} \frac{|F|\log(n+1)}{n} = 0.$$

Since F was chosen arbitrarily, we can conclude that $h(id_G) = 0$.

In the next fact we collect the basic properties of the algebraic entropy. The properties in (b), (c) and (d) were proved in [16] in the case of automorphisms and then generalized to endomorphisms in [5].

Fact 2.5. *Let (G, ϕ) be an algebraic flow.*

(a) *If H is a ϕ-invariant subgroup of G and $\overline{\phi} : G/H \to G/H$ is the endomorphism induced by ϕ, then*
$$h(\phi) \geq \max\{h(\phi \restriction_H), h(\overline{\phi})\}.$$
(b) *Let (H, η) be another algebraic flow. If ϕ and η are conjugated (i.e., there exists an isomorphism $\xi : G \to H$ such that $\phi = \xi^{-1}\eta\xi$), then $h(\phi) = h(\eta)$.*
(c) *If $G = G_1 \times G_2$ and $\phi_i \in \mathrm{End}(G_i)$, for $i = 1, 2$, then $h(\phi_1 \times \phi_2) = h(\phi_1) + h(\phi_2)$.*
(d) *If G is a direct limit of ϕ-invariant subgroups $\{G_i : i \in I\}$, then $h(\phi) = \sup_{i\in I} h(\phi \restriction_{G_i})$.*

Example 2.6. For any abelian group K, the *right Bernoulli shift* is $\beta_K : K^{(\mathbb{N})} \to K^{(\mathbb{N})}$ defined by

$$(x_0, x_1, x_2, \ldots) \mapsto (0, x_0, x_1, \ldots).$$

It is proved in [6] that $h(\beta_{\mathbb{Z}(p)}) = \mathrm{ent}(\beta_{\mathbb{Z}(p)}) = \log p$, where p is a prime. Then $h(\beta_{\mathbb{Z}}) = \infty$ in view of Fact 2.5(a). Therefore, $h(\beta_K) = \log|K|$ (see [5,6]), with the usual convention that $\log|K| = \infty$ if $|K|$ is infinite.

The following two useful properties of the algebraic entropy are needed in the proof of our main results, namely, Theorems 6.6 and 6.12.

Lemma 2.7. *Let (G, ϕ) be an algebraic flow. If G is torsion-free and $\widetilde{\phi} : D(G) \to D(G)$ denotes the unique extension of ϕ to the divisible hull $D(G)$ of G, then $h(\widetilde{\phi}) = h(\phi)$.*

Proof. It is obvious that $h(\widetilde{\phi}) \geq h(\phi)$ by Fact 2.5(a).

Let $F \in [D(G)]^{<\omega}$. Then there exists $m \in \mathbb{N}_+$ such that $mF \subseteq G$. Let $\mu_m(x) = mx$ for every $x \in D(G)$. Then μ_m is an automorphism of $D(G)$ that commutes with $\widetilde{\phi}$. Moreover, $T_n(\phi, mF) = T_n(\phi, \mu_m(F)) = \mu_m(T_n(\widetilde{\phi}, F))$. In particular, $\tau_{\phi,mF}(n) = \tau_{\widetilde{\phi},F}(n)$. Hence, $H(\phi, mF) = H(\widetilde{\phi}, F)$. Since $H(\phi, mF) \leq h(\phi)$, we conclude that $H(\widetilde{\phi}, F) \leq h(\phi)$. By the arbitrariness of F, this gives $h(\widetilde{\phi}) \leq h(\phi)$. □

Lemma 2.8. *Let (G, ϕ) be an algebraic flow and $F \in [G]^{<\omega}$. If $H(\phi, F) = 0$, then $h(\phi \restriction_{V(\phi,F)}) = 0$.*

Proof. Since $H(\phi, F) = 0$, we have $H(\phi, -F \cup \{0\} \cup F) = 0$ as well by Claim 2.1(e). Moreover, $V(\phi, F) \subseteq V(\phi, -F \cup \{0\} \cup F)$. So we can assume without loss of generality that $0 \in F$ and $F = -F$ by Fact 2.5(a). From $H(\phi, F) = 0$, it follows that $H(\phi, F_{(m)}) = 0$ for every $m \in \mathbb{N}_+$; indeed, fixed $m \in \mathbb{N}_+$, for every $n \in \mathbb{N}_+$, we have $T_n(\phi, F_{(m)}) = T_n(\phi, F)_{(m)}$, and so $\tau_{\phi,F_{(m)}}(n) \leq \tau_{\phi,F}(n)^m$.

Note that

$$V(\phi, F) = \bigcup_{m\in\mathbb{N}_+} T_m(\phi, F_{(m)}). \tag{2.1}$$

Moreover, for $m \in \mathbb{N}_+$,

$$H(\phi, F_{(m^2)}) = 0 \quad \text{implies} \quad H(\phi, T_m(\phi, F_{(m)})) = 0, \tag{2.2}$$

since $T_n(\phi, T_m(\phi, F_{(m)})) \subseteq T_{n+m}(\phi, F_{(m^2)})$ for every $n \in \mathbb{N}_+$. Now, for every $F' \in [V(\phi, F)]^{<\omega}$, by (2.1) there exists $m \in \mathbb{N}_+$ such that $F' \subseteq T_m(\phi, F_{(m)})$. By (2.2) $H(\phi, T_m(\phi, F_{(m)})) = 0$ and so also $H(\phi, F') = 0$. By the arbitrariness of F', this proves that $h(\phi \restriction_{V(\phi,F)}) = 0$. □

3. The Pinsker subgroup

The next proposition proves the existence of the Pinsker subgroup for any algebraic flow.

Proposition 3.1. *Let (G, ϕ) be an algebraic flow. The Pinsker subgroup $\mathbf{P}(G, \phi)$ of G exists.*

Proof. Let $\mathcal{F} = \{H \leq G : H\ \phi\text{-invariant}, h(\phi \restriction_H) = 0\}$.

We start proving that

$$\text{if } H_1, \ldots, H_n \in \mathcal{F}, \quad \text{then } H_1 + \cdots + H_n \in \mathcal{F}. \tag{3.1}$$

Let $H_1, H_2 \in \mathcal{F}$. Consider $\xi = \phi \restriction_{H_1} \times \phi \restriction_{H_2} : H_1 \times H_2 \to H_1 \times H_2$. By Fact 2.5(c) $h(\xi) = 0$. Since $H_1 + H_2$ is a quotient of $H_1 \times H_2$, the quotient endomorphism $\overline{\xi} : H_1 + H_2 \to H_1 + H_2$ induced by ξ has $h(\overline{\xi}) = 0$ by Fact 2.5(a). Since $\phi \restriction_{H_1+H_2}$ is conjugated to $\overline{\xi}$, it follows that $h(\phi \restriction_{H_1+H_2}) = 0$ by Fact 2.5(b). Proceeding by induction it is clear how to prove (3.1).

Let $P = \langle H : H \in \mathcal{F}\rangle$, and let $F \in [P]^{<\omega}$. Then there exist $n \in \mathbb{N}_+$ and $H_1, \ldots, H_n \in \mathcal{F}$ such that $F \subseteq H_1 + \cdots + H_n$. By (3.1) $h(\phi \restriction_{H_1+\cdots+H_n}) = 0$, so in particular $H(\phi \restriction_P, F) = H(\phi \restriction_{H_1+\cdots+H_n}, F) = 0$. Since F is arbitrary, this proves that $h(\phi \restriction_P) = 0$. By the definition of P, if H is a ϕ-invariant subgroup of G with $h(\phi \restriction_H) = 0$, then $H \subseteq P$. Hence $P = \mathbf{P}(G, \phi)$. □

It is clear that for an algebraic flow (G, ϕ), $h(\phi) = 0$ if and only if $G = \mathbf{P}(G, \phi)$. In the opposite direction we consider the following property.

Definition 3.2. Let (G, ϕ) be an algebraic flow. We say the ϕ has *completely positive algebraic entropy* if $h(\phi\restriction_H) > 0$ for every non-trivial ϕ-invariant subgroup H of G. We denote this by $h(\phi) \gg 0$.

Clearly, $h(\phi) \gg 0$ if and only if $\mathbf{P}(G, \phi) = 0$.

The definition is motivated by its topological counterpart: a topological flow (X, ψ) has *completely positive topological entropy* if all its non-trivial factors have positive topological entropy [4].

Lemma 3.3. *Let (G, ϕ) be an algebraic flow and let H be a ϕ-invariant subgroup of G. Then:*
(a) $\mathbf{P}(H, \phi\restriction_H) = \mathbf{P}(G, \phi) \cap H$;
(b) *for $\pi : G \to G/H$ the canonical projection and $\overline{\phi} : G/H \to G/H$ the endomorphism induced by ϕ, $\pi(\mathbf{P}(G, \phi)) \subseteq \mathbf{P}(G/H, \overline{\phi})$.*

Proof. (a) Since $h(\phi\restriction_{\mathbf{P}(H,\phi\restriction_H)}) = 0$, it follows that $\mathbf{P}(H, \phi\restriction_H) \subseteq \mathbf{P}(G, \phi) \cap H$. Since $\mathbf{P}(G, \phi) \cap H \subseteq \mathbf{P}(G, \phi)$, we have $h(\phi\restriction_{\mathbf{P}(G,\phi)\cap H}) = 0$ and so $\mathbf{P}(G, \phi) \cap H \subseteq \mathbf{P}(H, \phi\restriction_H)$.

(b) Follows immediately from Fact 2.5(a). $\square$

The inclusion in item (b) of the above lemma cannot be replaced by equality (e.g., if $G = \mathbb{Q}$, $H = \mathbb{Z}$ and ϕ is defined by $\phi(x) = 2x$ for $x \in \mathbb{Q}$, then $\mathbf{P}(G, \phi) = 0$, while $\mathbf{P}(G/H, \overline{\phi}) = G/H$).

Lemma 3.3 shows, in particular, the stability properties of the class of endomorphisms with zero algebraic entropy under taking invariant subgroups and quotients over invariant subgroups. By Fact 2.5(c) this class is preserved also under taking finite direct products, and so also under taking arbitrary direct sums. Indeed, if $G = \bigoplus_{i\in I} G_i$, then G is a direct limit of $\bigoplus_{i\in F} G_i$ where F runs over all non-empty finite subsets of I, and so Fact 2.5(d) applies. Example 3.5 below shows that the class of endomorphisms with zero algebraic entropy is not preserved under taking arbitrary infinite direct products.

Lemma 3.4 ([6]). *Let (G, ϕ) be an algebraic flow and assume that G is torsion. Then:*
(a) $h(\phi\restriction_{t_\phi(G)}) = 0$ *and* $t_{\overline{\phi}}(G/t_\phi(G)) = 0$ *for the endomorphism* $\overline{\phi} : G/t_\phi(G) \to G/t_\phi(G)$ *induced by* ϕ;
(b) *if H is a ϕ-invariant subgroup of G and $h(\phi\restriction_H) = 0$, then $H \subseteq t_\phi(G)$.*

Example 3.5. For a prime p, let $G = \prod_{n\in\mathbb{N}_+} \mathbb{Z}(p)^n$ and for every $n \in \mathbb{N}_+$ consider the right shift $\beta_n : \mathbb{Z}(p)^n \to \mathbb{Z}(p)^n$ defined by $(x_1, x_2, \dots, x_n) \mapsto (0, x_1, \dots, x_{n-1})$. Since $\mathbb{Z}(p)^n$ is finite, $h(\beta_n) = \mathrm{ent}(\beta_n) = 0$. On the other hand, let $\phi = \prod_{n\in\mathbb{N}_+} \beta_n : G \to G$; then $h(\phi) = \mathrm{ent}(\phi) > 0$. Indeed, for $n \in \mathbb{N}_+$, let $x_n \in \mathbb{Z}(p)^n$ be such that $\beta_{\mathbb{Z}(p)}^{n-1}(x_n) \neq 0$, hence $x = (x_n)_{n\in\mathbb{N}_+} \in G$ has infinite trajectory under ϕ. By Lemma 3.4(b), we have $h(\phi) = \mathrm{ent}(\phi) > 0$.

Let us see now that the algebraic flow (G, ϕ) fails to have a largest factor with zero algebraic entropy. Indeed, such a factor corresponds to a smallest ϕ-invariant subgroup H of G such that the induced endomorphism $\overline{\phi} : G/H \to G/H$ has zero algebraic entropy. For $m \in \mathbb{N}$ let $G_m = \prod_{n>m} \mathbb{Z}(p)^n$. Then G_m is a ϕ-invariant subgroup of G and the induced endomorphism $\overline{\phi} : G/G_m \to G/G_m$ has zero algebraic entropy (as G/G_m is finite). Then G_m contains H by the definition of H. Hence $H \subseteq \bigcap_{m\in\mathbb{N}} G_m = 0$. Therefore, $H = 0$, a contradiction (as $h(\phi) > 0$).

From items (a) and (b) of Lemma 3.4, we also obtain immediately the following.

Corollary 3.6. *Let (G, ϕ) be an algebraic flow. If G is torsion, then $\mathbf{P}(G, \phi) = t_\phi(G)$.*

Let (G, ϕ) be an algebraic flow. A point $x \in G$ is a *periodic* point of ϕ if there exists $n \in \mathbb{N}_+$ such that $\phi^n(x) = x$. Let $P_1(G, \phi)$ be the subset of G of all periodic points of ϕ. Obviously, $P_1(G, \phi)$ is a ϕ-invariant subgroup of G. Analogously, let $Q_1(G, \phi)$ be the subset of G of all quasi-periodic points of ϕ. Hence, $Q_1(G, \phi)$ is a ϕ-invariant subgroup of G as well, since $Q_1(G, \phi) = \bigcup_{n\in\mathbb{N}} \phi^{-n}(P_1(G, \phi))$.

Clearly, the quasi-periodic points of an injective endomorphism ϕ are periodic, that is, $Q_1(G, \phi) = P_1(G, \phi)$.

If $P_1(G, \phi) = G$, we say that ϕ is *locally periodic*, i.e., for every $x \in G$ there exists $n \in \mathbb{N}_+$ such that $\phi^n(x) = x$. Moreover, ϕ is *periodic* if there exists $n \in \mathbb{N}_+$ such that $\phi^n(x) = x$ for every $x \in G$. Analogously, if $Q_1(G, \phi) = G$, we say that ϕ is *locally quasi-periodic*, i.e., for every $x \in G$ there exist $n > m$ in $\mathbb{N}$ such that $\phi^n(x) = \phi^m(x)$. Finally, ϕ is *quasi-periodic* if there exist $n > m$ in $\mathbb{N}$ such that $\phi^n(x) = \phi^m(x)$ for every $x \in G$.

Proposition 3.7. *Let (G, ϕ) be an algebraic flow. Then*

$$t_\phi(G) = t(G) \cap Q_1(G, \phi) \subseteq Q_1(G, \phi) \subseteq \mathbf{P}(G, \phi).$$

If G is torsion, then all these three subgroups coincide.

Proof. The inclusion $t_\phi(G) \subseteq t(G)$ is obvious. If $x \in t_\phi(G)$, then $V(\phi, x)$ is a finite ϕ-invariant subgroup of G and so $V(\phi, x) \subseteq Q_1(G, \phi)$, hence $t_\phi(G) \subseteq Q_1(G, \phi)$. Since every quasi-periodic point has finite trajectory, for $x \in t(G) \cap Q_1(G, \phi)$ one has $x \in t_\phi(G)$. This proves the first equality. Moreover, $Q_1(G, \phi) \subseteq \mathbf{P}(G, \phi)$, since for every $F \in [G]^{<\omega}$ with $F \subseteq Q_1(G, \phi)$, there exists $m \in \mathbb{N}_+$ such that $T_n(\phi, F) = T_m(\phi, F)$ for every $n \in \mathbb{N}$, $n \geq m$, hence $H(\phi, F) = 0$. This proves that $h(\phi\restriction_{Q_1(G,\phi)}) = 0$, and so $Q_1(G, \phi) \subseteq \mathbf{P}(G, \phi)$.

If G is torsion, $\mathbf{P}(G, \phi) = t_\phi(G)$ by Corollary 3.6. $\square$

Note that always $t_{id_G}(G) = t(G) \subseteq Q_1(G, id_G) \subseteq \mathbf{P}(G, id_G)$.

The example of torsion abelian groups gives the motivation and the idea on how to approach the Pinsker subgroup of arbitrary abelian groups making use of quasi-periodic points. To this end we need a generalization of the notions of periodic and quasi-periodic points.

4. Generalized quasi-periodic points

Let (G, ϕ) be an algebraic flow. We extend the definition of $P_1(G, \phi)$ setting by induction:

(a) $P_0(G, \phi) = 0$, and for every $n \in \mathbb{N}$
(b) $P_{n+1}(G, \phi) = \{x \in G : (\exists n \in \mathbb{N}_+)\ \phi^n(x) - x \in P_n(G, \phi)\}$.

This gives an increasing chain

$$P_0(G, \phi) \subseteq P_1(G, \phi) \subseteq \cdots \subseteq P_n(G, \phi) \subseteq \cdots .$$

We show below that all members of this chain are ϕ-invariant subgroups of G. Our interest in these subgroups is motivated by the fact that they are contained in $\mathbf{P}(G, \phi)$.

In order to approximate better $\mathbf{P}(G, \phi)$ we need to enlarge these subsets introducing for every $n \in \mathbb{N}$ appropriate counterparts of $Q_1(G, \phi)$, as follows. Define

(a) $Q_0(G, \phi) = 0$, and for every $n \in \mathbb{N}$
(b) $Q_{n+1}(G, \phi) = \{x \in G : (\exists n > m \text{ in } \mathbb{N})\ (\phi^n - \phi^m)(x) \in Q_n(G, \phi)\}$.

We get an increasing chain

$$Q_0(G, \phi) \subseteq Q_1(G, \phi) \subseteq \cdots \subseteq Q_n(G, \phi) \subseteq \cdots . \tag{4.1}$$

Again we show below that the members of this chain are ϕ-invariant subgroups of G. One can prove by induction that for injective endomorphisms ϕ of an abelian group G, $Q_n(G, \phi) = P_n(G, \phi)$ for every $n \in \mathbb{N}$.

Definition 4.1. For an algebraic flow (G, ϕ), let $\mathfrak{Q}(G, \phi) = \bigcup_{n\in\mathbb{N}} Q_n(G, \phi)$.

Proposition 4.2. *Let (G, ϕ) be an algebraic flow.*

(a) *For every $n \in \mathbb{N}$, $P_n(G, \phi)$ is a ϕ-invariant subgroup of G, and for the induced endomorphism $\overline{\phi}_n : G/P_n(G, \phi) \to G/P_n(G, \phi)$ and the canonical projection $\pi_n : G \to G/P_n(G, \phi)$,*

$$P_{n+1}(G, \phi) = \pi_n^{-1}(P_1(G/P_n(G, \phi), \overline{\phi}_n)) \quad (\text{i.e., } P_{n+1}(G, \phi)/P_n(G, \phi) = P_1(G/P_n(G, \phi), \overline{\phi}_n)). \tag{4.2}$$

(b) *For every $n \in \mathbb{N}$, $Q_n(G, \phi)$ is a ϕ-invariant subgroup of G, and for the induced endomorphism $\overline{\phi}_n : G/Q_n(G, \phi) \to G/Q_n(G, \phi)$ and the canonical projection $\pi_n : G \to G/Q_n(G, \phi)$,*

$$Q_{n+1}(G, \phi) = \pi_n^{-1}(Q_1(G/Q_n(G, \phi), \overline{\phi}_n)) \quad (\text{i.e., } Q_{n+1}(G, \phi)/Q_n(G, \phi) = Q_1(G/Q_n(G, \phi), \overline{\phi}_n)). \tag{4.3}$$

(c) *$\mathfrak{Q}(G, \phi)$ is a ϕ-invariant subgroup of G.*

Proof. (a) We proceed by induction. The cases $n = 0$ and $n = 1$ are trivial. Let $n \in \mathbb{N}$ and assume that $P_n(G, \phi)$ is a ϕ-invariant subgroup of G.

Let $x \in P_{n+1}(G, \phi)$. This is equivalent to the existence of $k \in \mathbb{N}_+$ such that $\phi^k(x) - x \in P_n(G, \phi)$. This occurs if and only if $\pi_n(\phi^k(x) - x) = 0$ in $G/P_n(G, \phi)$. Since $\overline{\phi}_n^k(\pi_n(x)) - \pi_n(x) = \pi_n(\phi^k(x) - x)$, this is equivalent to $\pi_n(x) \in P_1(G/P_n(G, \phi), \overline{\phi}_n)$, that is, $x \in \pi_n^{-1}(P_1(G/P_n(G, \phi), \overline{\phi}_n))$. So this proves (4.2).

Since $P_1(G/P_n(G, \phi), \overline{\phi}_n)$ is a $\overline{\phi}_n$-invariant subgroup of $G/P_n(G, \phi)$, its counterimage $P_{n+1}(G, \phi)$ is a ϕ-invariant subgroup of G.

(b) Argue as in (a).

(c) By (a), (b) and its definition, $\mathfrak{Q}(G, \phi)$ is a ϕ-invariant subgroup of G. □

It is worth noting the following easy to prove properties.

Lemma 4.3. *Let (G, ϕ) be an algebraic flow. Then:*

(a) *$P_n(H, \phi\restriction_H) = P_n(G, \phi) \cap H$ and $Q_n(H, \phi\restriction_H) = Q_n(G, \phi) \cap H$ for every $n \in \mathbb{N}$; so*
(b) *$\mathfrak{Q}(H, \phi\restriction_H) = \mathfrak{Q}(G, \phi) \cap H$.*
(c) *Moreover, $\mathfrak{Q}(G, \phi) = 0$ if and only if $Q_1(G, \phi) = 0$.*

In Example 4.4 we discuss the length of the chain (4.1) for an algebraic flow (G, ϕ). We shall prove in Proposition 4.11 that the endomorphism $\overline{\phi} : G/\mathfrak{Q}(G, \phi) \to G/\mathfrak{Q}(G, \phi)$ induced by ϕ satisfies $\mathfrak{Q}(G/\mathfrak{Q}(G, \phi), \overline{\phi}) = 0$. In other words, the chain (4.1) cannot be extended by adding new terms. This justifies the following definition: for $n \in \mathbb{N}$, we say that the *Loewy length* of $\mathfrak{Q}(G, \phi)$ is n, if the chain (4.1) stabilizes at the term n, otherwise we say that the Loewy length of $\mathfrak{Q}(G, \phi)$ is ω.

Example 4.4. For an algebraic flow (G, ϕ) the Loewy length of $\mathfrak{Q}(G, \phi)$ may be arbitrarily large (up to ω). In item (a) we point out a large class of examples when the length is 1. In items (b) and (c) we give examples of length 2 and ω, respectively. With similar examples one can show that it may take all values from 0 to ω.

(a) If G is torsion, then by Proposition 3.7 $\mathfrak{Q}(G, \phi) = t_\phi(G)$ coincides with the Pinsker subgroup. To see that $\mathfrak{Q}(G, \phi) = Q_1(G, \phi)$ has length 1 it suffices to recall that $t_\phi(G/t_\phi(G)) = 0$ according to [6, Lemma 2.3(1)].

(b) Let $G = \mathbb{Z}^2$ and let ϕ be the automorphism of G defined by $\phi(x, y) = (x + y, y)$, that is, by the matrix $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$. The subgroup $H = \mathbb{Z} \times \{0\}$ is ϕ-invariant. Moreover, $Q_1(G, \phi) = H$, while $\mathfrak{Q}(G, \phi) = Q_2(G, \phi) = G$. Consequently,

$$0 = Q_0(G, \phi) \subset Q_1(G, \phi) \subset Q_2(G, \phi) = \mathfrak{Q}(G, \phi) = G.$$

(c) Let $G = \mathbb{Z}^{(\mathbb{N}_+)} = \bigoplus_{n=1}^{\infty} \langle e_n \rangle$. Let ϕ be the automorphism of G given by the matrix

$$\begin{pmatrix}1 & 1 & 1 & \dots\\ 0 & 1 & 1 & \dots\\ 0 & 0 & 1 & \dots\\ \vdots & \vdots & \ddots & \ddots\end{pmatrix}.$$

For every $n \in \mathbb{N}_+$ let $G_n = \langle e_1, \dots, e_n \rangle$. Then $G_n = P_n(G, \phi) = Q_n(G, \phi)$ and $\mathfrak{Q}(G, \phi) = G$. In particular, we have the following strictly increasing chain

$$0 = Q_0(G, \phi) \subset Q_1(G, \phi) \subset Q_2(G, \phi) \subset \cdots \subset Q_n(G, \phi) \subset \cdots \subset \mathfrak{Q}(G, \phi) = G.$$

Recall that a subgroup H of a torsion-free abelian group G is *pure* in G if and only if G/H is torsion-free.

Lemma 4.5. *Let (G, ϕ) be an algebraic flow. If G is torsion-free, then $Q_n(G, \phi)$, for every $n \in \mathbb{N}$, and $\mathfrak{Q}(G, \phi)$ are pure in G.*

Proof. We proceed by induction. The case $n = 0$ is trivial.

Let $k \in \mathbb{N}$ and $x \in Q_1(G, \phi) \cap kG$. Then $x = ky$ for some $y \in G$ and there exist $n > m$ in $\mathbb{N}$ such that $\phi^n(x) = \phi^m(x)$. Consequently, $k\phi^n(y) = k\phi^m(y)$, which yields $\phi^n(y) = \phi^m(y)$ as G is torsion-free. So $y \in Q_1(G, \phi)$, i.e., $x \in kQ_1(G, \phi)$. This proves that $Q_1(G, \phi)$ is pure in G.

In the sequel we let $G_m = G/Q_m(G, \phi)$ for $m \in \mathbb{N}$. Assume that $Q_n(G, \phi)$ is pure in G (so G_n is torsion-free) for some $n \in \mathbb{N}$. We have to show that G_{n+1} is torsion-free as well. Let $\overline{\phi}_n : G_n \to G_n$ be the endomorphism induced by ϕ. Then $G_{n+1} = G/Q_{n+1}(G, \phi) \cong G_n/(Q_{n+1}(G, \phi)/Q_n(G, \phi)) = G_n/Q_1(G_n, \overline{\phi}_n)$, where the last equality holds in view of (4.3). By the case $n = 1$, applied to the torsion-free group G_n, the latter quotient is torsion-free, so G_{n+1} is torsion-free as well.

As an increasing union of pure subgroups, $\mathfrak{Q}(G, \phi)$ is pure itself. □

Claim 4.6. *Let (G, ϕ) be an algebraic flow, H a ϕ-invariant subgroup of G and $x \in G$. If $\phi^k(x) - x \in H$ for some $k \in \mathbb{N}_+$, then $\phi^{nk}(x) - x \in H$ for every $n \in \mathbb{N}_+$. In particular, $P_n(G, \phi) = P_n(G, \phi^m)$ for every $n \in \mathbb{N}$ and every $m \in \mathbb{N}_+$.*

Proof. Let $n \in \mathbb{N}_+$, let $\overline{\phi} : G/H \to G/H$ be the endomorphism induced by ϕ, and $\pi : G \to G/H$ the canonical projection. It suffices to note that $\phi^m(x) - x \in H$ for some $m \in \mathbb{N}_+$ if and only if $\overline{\phi}^m(\pi(x)) = \pi(x)$, so one can argue by induction on n. □

Note that $\ker_\infty \phi \subseteq Q_1(G, \phi)$, so also $\ker_\infty \phi \subseteq Q_n(G, \phi)$ for every $n \in \mathbb{N}_+$, and in particular $\ker_\infty \phi \subseteq \mathfrak{Q}(G, \phi)$.

Lemma 4.7. *Let (G, ϕ) be an algebraic flow. Then $Q_n(G, \phi) = P_n(G, \phi) \oplus \ker_\infty \phi$ for every $n \in \mathbb{N}_+$.*

Proof. We start proving that $P_n(G, \phi) \cap \ker_\infty \phi = 0$, proceeding by induction. For $n = 1$, let $x \in P_1(G, \phi) \cap \ker_\infty \phi$. Then there exist $s, t \in \mathbb{N}_+$ such that $\phi^s(x) = x$ and $\phi^t(x) = 0$. Consequently, $x = \phi^{st}(x) = 0$. Assume now that $n \in \mathbb{N}_+$ and that $P_n(G, \phi) \cap \ker_\infty \phi = 0$. Let $x \in P_{n+1}(G, \phi) \cap \ker_\infty \phi$. Then there exist $s, t \in \mathbb{N}_+$ such that $\phi^s(x) - x \in P_n(G, \phi)$ and $\phi^t(x) = 0$. By Claim 4.6, $\phi^{st}(x) - x \in P_n(G, \phi)$. Since $\phi^{st}(x) = 0$, this yields $x \in P_n(G, \phi) \cap \ker_\infty \phi$. By the inductive hypothesis $x = 0$.

For the quotient $G/\ker_\infty \phi$ and the canonical projection $\pi : G \to G/\ker_\infty \phi$, the induced endomorphism $\overline{\phi} : G/\ker_\infty \phi \to G/\ker_\infty \phi$ is injective. Hence

$$Q_n(G/\ker_\infty \phi, \overline{\phi}) = P_n(G/\ker_\infty \phi, \overline{\phi}) \tag{4.4}$$

for every $n \in \mathbb{N}$. Then, in order to show that $Q_n(G, \phi) = P_n(G, \phi) + \ker_\infty \phi$ for every $n \in \mathbb{N}_+$, it suffices to show that for every $n \in \mathbb{N}$:

(a) $\pi(P_n(G, \phi)) = P_n(G/\ker_\infty \phi, \overline{\phi})$, and
(b) $\pi(Q_n(G, \phi)) = Q_n(G/\ker_\infty \phi, \overline{\phi})$.

(a) Clearly $\pi(P_n(G, \phi)) \subseteq P_n(G/\ker_\infty \phi, \overline{\phi})$ for every $n \in \mathbb{N}$. So we prove by induction the converse inclusion. The case $n = 0$ is trivial. Assume now that $n \in \mathbb{N}$ and that $\pi(P_n(G, \phi)) = P_n(G/\ker_\infty \phi, \overline{\phi})$ holds true. Let $x \in G$ be such that $\pi(x) \in P_{n+1}(G/\ker_\infty \phi, \overline{\phi})$. Then there exists $s \in \mathbb{N}_+$ such that $\overline{\phi}^s(\pi(x)) - \pi(x) \in P_n(G/\ker_\infty \phi, \overline{\phi})$. Since $\pi(\phi^s(x) - x) = \overline{\phi}^s(\pi(x)) - \pi(x) \in P_n(G/\ker_\infty \phi, \overline{\phi})$, we conclude that

$$\phi^s(x) - x \in P_n(G, \phi) + \ker_\infty \phi. \tag{4.5}$$

Hence there exists $k \in \mathbb{N}_+$ such that $\phi^k(\phi^s(x) - x) \in P_n(G, \phi)$ and so $\phi^s(\phi^k(x)) - \phi^k(x) \in P_n(G, \phi)$, that is, $\phi^k(x) \in P_{n+1}(G, \phi)$. By Claim 4.6 applied to (4.5), and since $P_{n+1}(G, \phi)$ is ϕ-invariant,

$$\phi^{sk}(x) - x \in P_n(G, \phi) + \ker_\infty \phi \quad \text{and} \quad \phi^{sk}(x) \in P_{n+1}(G, \phi).$$

Hence $x \in \ker_\infty \phi + P_{n+1}(G, \phi)$, and consequently $\pi(x) \in \pi(P_{n+1}(G, \phi))$.

(b) Let $n \in \mathbb{N}$. Clearly $\pi(P_n(G,\phi)) \subseteq \pi(Q_n(G,\phi)) \subseteq Q_n(G/\ker_\infty \phi, \overline{\phi})$. Moreover, $Q_n(G/\ker_\infty \phi, \overline{\phi}) = P_n(G/\ker_\infty \phi, \overline{\phi}) = \pi(P_n(G,\phi))$, by (4.4) and (a). Hence all these subgroups coincide, and in particular $\pi(Q_n(G,\phi)) = Q_n(G/\ker_\infty \phi, \overline{\phi})$. □

It is easy to see that if G is an abelian group and $\phi \in \mathrm{Aut}(G)$, then $\phi(Q_n(G,\phi)) = Q_n(G,\phi)$ and $Q_n(G,\phi^{-1}) = Q_n(G,\phi)$ for every $n \in \mathbb{N}$. Consequently, $\phi(\mathfrak{Q}(G,\phi)) = \mathfrak{Q}(G,\phi)$ and $\mathfrak{Q}(G,\phi^{-1}) = \mathfrak{Q}(G,\phi)$.

Proposition 4.8. *Let (G,ϕ) be an algebraic flow. Then $Q_n(G,\phi) = \phi^{-1}(Q_n(G,\phi))$ (i.e., the induced endomorphism $\overline{\phi}_n : G/Q_n(G,\phi) \to G/Q_n(G,\phi)$ is injective) for every $n \in \mathbb{N}$. Hence, $\mathfrak{Q}(G,\phi) = \phi^{-1}(\mathfrak{Q}(G,\phi))$ (i.e., the induced endomorphism $G/\mathfrak{Q}(G,\phi) \to G/\mathfrak{Q}(G,\phi)$ is injective).*

Proof. The equality $Q_1(G,\phi) = \phi^{-1}(Q_1(G,\phi))$ easily follows from the definitions. Then an inductive argument using (4.3) applies. □

From Proposition 4.8 one easily obtains:

Corollary 4.9. *If (G,ϕ) is an algebraic flow with surjective ϕ, then the induced endomorphisms $G/\mathfrak{Q}(G,\phi) \to G/\mathfrak{Q}(G,\phi)$ and $G/Q_n(G,\phi) \to G/Q_n(G,\phi)$ ($n \in \mathbb{N}$) are automorphisms.*

The following notion is motivated by its connection to ergodic theory (an automorphism ϕ of an abelian group G is algebraically ergodic if and only if its Pontryagin dual $\widehat{\phi}$ is an ergodic automorphism of the compact abelian group $\widehat{G}$ [21]).

Definition 4.10. Let (G,ϕ) be an algebraic flow. Call ϕ *algebraically ergodic* if $\mathfrak{Q}(G,\phi) = 0$ (that is, ϕ has no non-trivial quasi-periodic point).

Observe that for an abelian group G, the endomorphism 0_G is algebraically ergodic if and only if $G = 0$.

The next proposition shows that, for (G,ϕ) an algebraic flow, $\mathfrak{Q}(G,\phi)$ is the smallest ϕ-invariant subgroup H of G such that the induced endomorphism $\overline{\phi} : G/H \to G/H$ is algebraically ergodic.

Proposition 4.11. *Let (G,ϕ) be an algebraic flow. Then:*

(a) *the endomorphism $\overline{\phi} : G/\mathfrak{Q}(G,\phi) \to G/\mathfrak{Q}(G,\phi)$ induced by ϕ is algebraically ergodic, i.e.,*

$$\mathfrak{Q}(G/\mathfrak{Q}(G,\phi), \overline{\phi}) = 0;$$

(b) *if for some ϕ-invariant subgroup H of G the endomorphism $\overline{\phi} : G/H \to G/H$ induced by ϕ is algebraically ergodic, then $H \supseteq \mathfrak{Q}(G,\phi)$.*

Proof. (a) According to Proposition 4.8 it suffices to check that $\overline{\phi}$ has no non-zero periodic points. Let $\pi : G \to G/\mathfrak{Q}(G,\phi)$ be the canonical projection, and assume that $\pi(x) \in G/\mathfrak{Q}(G,\phi)$ is a periodic point of $\overline{\phi}$ for some $x \in G$. Then there exists $n \in \mathbb{N}_+$ such that $\phi^n(x) - x \in \mathfrak{Q}(G,\phi)$. Consequently $\phi^n(x) - x \in Q_s(G,\phi)$ for some $s \in \mathbb{N}$. This yields $x \in Q_{s+1}(G,\phi) \subseteq \mathfrak{Q}(G,\phi)$. Thus $\pi(x) = 0$ in $G/\mathfrak{Q}(G,\phi)$. Hence $Q_1(G/\mathfrak{Q}(G,\phi), \overline{\phi}) = 0$. By Lemma 4.3(c), $\mathfrak{Q}(G/\mathfrak{Q}(G,\phi), \overline{\phi}) = 0$.

(b) It suffices to see that $Q_n(G,\phi) \subseteq H$ for each $n \in \mathbb{N}$. We shall prove it by induction on $n \in \mathbb{N}$, the case $n = 0$ being trivial. If $n \in \mathbb{N}$ and $Q_n(G,\phi) \subseteq H$, then for $x \in Q_{n+1}(G,\phi)$ and $\pi : G \to G/H$ the canonical projection, $\pi(x) \in G/H$ is a quasi-periodic point of the induced endomorphism $\overline{\phi} : G/H \to G/H$, as $Q_{n+1}(G,\phi)/Q_n(G,\phi) = Q_1(G/Q_n(G,\phi), \overline{\phi}_n)$ by (4.3), where $\overline{\phi}_n : G/Q_n(G,\phi) \to G/Q_n(G,\phi)$ is the induced endomorphism. So our hypothesis yields $\pi(x) = 0$, that is, $x \in H$. □

5. The polynomial growth

Example 5.1. Let G be an abelian group.

(a) Then $id_G \in \mathrm{Pol}$ by Claim 2.2 and it is obvious that $0_G \in \mathrm{Pol}$.
(b) Moreover, $\phi \restriction_{\ker_\infty \phi} \in \mathrm{Pol}$. Indeed, for every $F \in [\ker_\infty \phi]^{<\omega}$, there exists $m \in \mathbb{N}_+$ such that $\phi^m(F) = 0$. Then $\tau_F(n) = \tau_F(m)$ for every $n \in \mathbb{N}$ with $n \geq m$. In particular, $\phi \in \mathrm{Pol}_F$.

The following obvious claim underlines the fact that the property of having polynomial growth is in some sense "local".

Claim 5.2. *Let (G,ϕ) be an algebraic flow and $F \in [G]^{<\omega}$. The following conditions are equivalent:*

(a) *$\phi \in \mathrm{Pol}_F$;*
(b) *$\phi \restriction_{V(\phi,F)} \in \mathrm{Pol}_F$;*
(c) *$\phi \restriction_H \in \mathrm{Pol}_F$ for every ϕ-invariant subgroup H of G such that $F \subseteq H$.*

Using the argument from the proof of Lemma 2.8, one can prove that the above equivalent conditions imply the stronger one $\phi \restriction_{V(\phi,F)} \in \mathrm{Pol}$. We are not giving this proof, since this stronger property can be deduced from Lemma 2.8 and Theorem 6.12.

The next proposition gives basic properties of endomorphisms with polynomial growth. These are analogous to the properties considered for the algebraic entropy (see Fact 2.5).

Proposition 5.3. *Let (G,ϕ) be an algebraic flow.*

(a) *Let H be a ϕ-invariant subgroup of G and $\overline{\phi}: G/H \to G/H$ the endomorphism induced by ϕ. If $\phi \in \mathrm{Pol}$, then $\phi\restriction_H \in \mathrm{Pol}$ and $\overline{\phi} \in \mathrm{Pol}$.*
(b) *Let (H,η) be another algebraic flow. If ϕ and η are conjugated, (i.e., there exists an isomorphism $\xi: G \to H$ such that $\phi = \xi^{-1}\eta\xi$), then $\phi \in \mathrm{Pol}$ if and only if $\eta \in \mathrm{Pol}$. More precisely, if $F' \in [H]^{<\omega}$, then $\eta \in \mathrm{Pol}_{F'}$ if and only if $\phi \in \mathrm{Pol}_{\xi^{-1}(F')}$ (if $F \in [G]^{<\omega}$, then $\phi \in \mathrm{Pol}_F$ if and only if $\eta \in \mathrm{Pol}_{\xi(F)}$).*
(c) *If $G = G_1 \times G_2$ and $\phi_i \in \mathrm{End}(G_i)$, for $i = 1,2$, then $\phi_1 \times \phi_2 \in \mathrm{Pol}$ if and only if $\phi_1 \in \mathrm{Pol}$ and $\phi_2 \in \mathrm{Pol}$.*
(d) *Let G be a direct limit of ϕ-invariant subgroups $\{G_i : i \in I\}$. If $\phi\restriction_{G_i} \in \mathrm{Pol}$ for every $i \in I$, then $\phi \in \mathrm{Pol}$.*
(e) *If G is torsion-free, then $\phi \in \mathrm{Pol}$ if and only if $\widetilde{\phi} \in \mathrm{Pol}$, where $D(G)$ is the divisible hull of G and $\widetilde{\phi}: D(G) \to D(G)$ is the unique extension of ϕ to $D(G)$.*

Proof. The proofs of (a), (b), (c) and (d) are straightforward, so we omit them.

(e) Let $F \in [D(G)]^{<\omega}$. Then there exists $m \in \mathbb{N}_+$ such that $mF \subseteq G$. Let $\mu_m(x) = mx$ for every $x \in D(G)$. Then μ_m is an automorphism of $D(G)$ that commutes with $\widetilde{\phi}$. Moreover, $T_n(\phi, mF) = T_n(\phi, \mu_m(F)) = \mu_m(T_n(\widetilde{\phi}, F))$. In particular, $\tau_{\widetilde{\phi},F} = \tau_{\phi,mF}$. Hence $\widetilde{\phi} \in \mathrm{Pol}_F$, as $\phi \in \mathrm{Pol}_{mF}$. □

The following property is related to powers of endomorphisms.

Lemma 5.4. *Let (G,ϕ) be an algebraic flow, let $k \in \mathbb{N}_+$ and $F \in [G]^{<\omega}$ with $0 \in F$.*

(a) *If $\phi \in \mathrm{Pol}_F$, then $\phi^k \in \mathrm{Pol}_F$.*
(b) *If $\phi^k \in \mathrm{Pol}_{T_k(\phi,F)}$, then $\phi \in \mathrm{Pol}_F$.*

In particular, $\phi \in \mathrm{Pol}$ if and only if $\phi^k \in \mathrm{Pol}$.

Proof. Let $n \in \mathbb{N}_+$. We can suppose without loss of generality that $0 \in F$ (so that the cardinality of the trajectories grows with n).

(a) Since $\phi \in \mathrm{Pol}_F$, there exists $P_F(X) \in \mathbb{Z}[X]$ such that $\tau_{\phi,F}(n) \le P_F(n)$ for every $n \in \mathbb{N}_+$. By Claim 2.1(b), for every $n \in \mathbb{N}_+$,

$$\tau_{\phi^k,F}(n) \le \tau_{\phi,F}(kn-k+1) \le P_F(kn-k+1).$$

This shows that $\phi^k \in \mathrm{Pol}_F$.

(b) Since $\phi^k \in \mathrm{Pol}_{T_k(\phi,F)}$, there exists a polynomial $P(X) \in \mathbb{Z}[X]$, depending only on F and the fixed k, such that $\tau_{\phi^k,T_k(\phi,F)}(n) \le P(n)$ for every $n \in \mathbb{N}_+$. By Claim 2.1(c), for every $n \in \mathbb{N}_+$,

$$\tau_{\phi,F}(n) \le \tau_{\phi,F}(nk) = \tau_{\phi^k,T_k(\phi,F)}(n) \le P(n).$$

This proves that $\phi \in \mathrm{Pol}_F$.

The last assertion follows directly from (a) and (b). □

Example 5.5. Let (G,ϕ) be an algebraic flow. If either $\phi^k = id_G$ (i.e., ϕ is periodic) or $\phi^k = 0_G$ (i.e., ϕ is nilpotent) for some $k \in \mathbb{N}_+$, then $\phi \in \mathrm{Pol}$. This follows immediately from Example 5.1 and Lemma 5.4.

Definition 5.6. For an algebraic flow (G,ϕ), let $\mathrm{Pol}(G,\phi)$ be the greatest ϕ-invariant subgroup of G such that $\phi\restriction_{\mathrm{Pol}(G,\phi)} \in \mathrm{Pol}$.

The proof of the following lemma is the same as the proof of the existence of the Pinsker subgroup of an algebraic flow given in Proposition 3.1, just replacing the properties in Fact 2.5 with those in Proposition 5.3.

Lemma 5.7. *Let (G,ϕ) be an algebraic flow. Then $\mathrm{Pol}(G,\phi)$ exists.*

For an algebraic flow (G,ϕ), one may ask whether $\mathrm{Pol}(G/\mathrm{Pol}(G,\phi),\overline{\phi}) = 0$, where $\overline{\phi}: G/\mathrm{Pol}(G,\phi) \to G/\mathrm{Pol}(G,\phi)$ is the endomorphism induced by ϕ. This property in fact holds true and will follow from results below (see Proposition 4.11 and Theorem 6.9).

Claim 5.8. *Let (G,ϕ) be an algebraic flow. If $G = V(\phi,F)$ for some $F \in [G]^{<\omega}$ and ϕ is locally periodic, then ϕ is periodic.*

Proof. There exists $m \in \mathbb{N}_+$ such that $\phi^m\restriction_F = id_F$. Hence $\phi^m = id_G$. □

The next claim generalizes Claim 2.2.

Claim 5.9. *Let (G,ϕ) be an algebraic flow and $F \in [G]^{<\omega}$. If $(\phi - id_G)^m = 0$ for some $m \in \mathbb{N}_+$, then $\tau_{\phi,F}(n) \le (n^m+1)^{m|F|}$ for every $n \in \mathbb{N}_+$.*

Proof. Let $|F| = t$, and let $s = \phi - id_G$, so that $s^m = 0$. Then

(1) $\phi^n(x) = x + C_n^1 s(x) + C_n^2 s^2(x) + \cdots + C_n^{m-1} s^{m-1}(x)$ for every $x \in G$ and $n \in \mathbb{N}$ with $n \ge m$; therefore
(2) $\phi^n(F) \subseteq F + s(F)_{(C_n^1)} + s^2(F)_{(C_n^2)} + \cdots + s^{m-1}(F)_{(C_n^{m-1})}$ for every $F \in [G]^{<\omega}$ with $0 \in F$ and every $n \in \mathbb{N}$ with $n \ge m$.

Hence

$$\phi^n(F) \subseteq F + s(F)_{(n)} + s^2(F)_{(n^2)} + \cdots + s^{m-1}(F)_{(n^{m-1})}$$

for every $F \in [G]^{<\omega}$ with $0 \in F$ and every $n \in \mathbb{N}$ with $n \geq m$. Therefore

(3) $T_n(\phi, F) \subseteq F_{(n)} + s(F)_{(n^2)} + s^2(F)_{(n^3)} + \cdots + s^{m-1}(F)_{(n^m)}$ for every $F \in [G]^{<\omega}$ with $0 \in F$ and every $n \in \mathbb{N}$ with $n \geq m$.

If $|F| = t$, then $|s^k(F)| \leq t$ for every $k \in \mathbb{N}$, so applying Claim 2.2 to each $s^k(F)$ and (3), we find

$$\begin{aligned}\tau_{\phi,F}(n) &\leq |F_{(n)}| \cdot |s(F)_{(n^2)}| \cdot |s^2(F)_{(n^3)}| \cdot \ldots \cdot |s^{m-1}(F)_{(n^m)}| \\ &\leq (n+1)^t \cdot (n^2+1)^t \cdot (n^3+1)^t \cdot \ldots \cdot (n^m+1)^t \leq (n^m+1)^{mt},\end{aligned}$$

as desired. □

Proposition 5.10. *Let (G, ϕ) be an algebraic flow. Then $\phi \restriction_{\mathfrak{Q}(G,\phi)} \in$ Pol.*

Proof. We start proving that

$$\phi \restriction_{P_m(G,\phi)} \in \mathrm{Pol} \quad \text{for every } m \in \mathbb{N}. \tag{5.1}$$

Fix $m \in \mathbb{N}$ and $F \in [P_m(G, \phi)]^{<\omega}$ with $0 \in F$. We want to prove that $\phi \in \mathrm{Pol}_F$, that is, we have to find $P_F(X) \in \mathbb{Z}[X]$ such that $\tau_{\phi,F}(n) \leq P_F(n)$ for every $n \in \mathbb{N}_+$. Since $V(\phi, F)$ is a ϕ-invariant subgroup of $P_m(G, \phi)$ and since

$$P_m(V(\phi, F), \phi \restriction_{V(\phi,F)}) = P_m(G, \phi) \cap V(\phi, F) = V(\phi, F)$$

in view of Lemma 4.3(a), by Claim 5.2 we can assume without loss of generality that $G = V(\phi, F)$.

For every $k \in \{0, \ldots, m-2\}$ let

$$\overline{G}_k = G/P_{m-k-1}(G, \phi) \quad \text{and} \quad G_k = P_{m-k}(G, \phi)/P_{m-k-1}(G, \phi).$$

Let $\overline{\phi}_k : \overline{G}_k \to \overline{G}_k$ be the endomorphism induced by ϕ. Then $\overline{\phi}_k \restriction_{G_k}: G_k \to G_k$ is locally periodic, because $G_k = P_1(\overline{G}_k, \overline{\phi}_k)$ by definition. Consider $\overline{G}_k$ as a $\mathbb{Z}[X]$-module letting $X \cdot g = \overline{\phi}_k(g)$ for every $g \in \overline{G}_k$. Let $\pi_k : G \to G/P_{m-k-1}(G, \phi)$ be the canonical projection. Our assumption $G = V(G, \phi)$ implies that the finite subset $\pi_k(F)$ generates $\overline{G}_k$ as a $\mathbb{Z}[X]$-module, i.e., $\overline{G}_k = V(\overline{\phi}_k, \pi_k(F))$. Since $\mathbb{Z}[X]$ is noetherian, the submodule G_k of $\overline{G}_k$ is finitely generated as well, that is, $G_k = V(\overline{\phi}_k, F_k)$ for some $F_k \in [G_k]^{<\omega}$. By Claim 5.8, $\overline{\phi}_k$ is periodic on G_k. Then there exists $w_k \in \mathbb{N}_+$ such that $\overline{\phi}_k^{w_k} \restriction_{G_k} = id_{G_k}$. Let $w = w_0 \cdot \ldots \cdot w_{m-2}$. Then $\overline{\phi}_k^{w} \restriction_{G_k} = id_{G_k}$.

By Claim 2.1(d), $G = V(\phi, F) = \langle T(\phi, F)\rangle = \langle T(\phi^w, T_w(\phi, F))\rangle$; moreover, $P_l(G, \phi) = P_l(G, \phi^w)$ for every $l \in \mathbb{N}_+$ by Claim 4.6. By Lemma 5.4(b), $\phi^w \in \mathrm{Pol}_{T_w(\phi,F)}$ implies $\phi \in \mathrm{Pol}_F$. So without loss of generality we can replace ϕ^w by ϕ, that is, we can suppose that $\overline{\phi}_k \restriction_{G_k} = id_{G_k}$.

Let $s = \phi - id_G$; then $s(P_{m-k}(G, \phi)) \subseteq P_{m-k-1}(G, \phi)$ for every $k \in \{0, \ldots, m-2\}$. In particular, $s^m = 0$. By Claim 5.9(b) $\tau_{\phi,F}(n) \leq (n^m+1)^t$ for every $n \in \mathbb{N}_+$ with $n \geq m$. Let $P_F(X) = (X^m+1)^t + k \in \mathbb{Z}[X]$, where $k = \tau_{\phi,F}(m)$. Then $\tau_{\phi,F}(n) \leq P_F(n)$ for every $n \in \mathbb{N}_+$, and this shows that $\phi \in \mathrm{Pol}_F$. Since F was chosen arbitrary, we have $\phi \in$ Pol. This concludes the proof of (5.1).

By Example 5.1(b), $\phi \restriction_{\ker_\infty \phi} \in$ Pol. Hence Proposition 5.3(c), (5.1) and Lemma 4.7 imply that $\phi \restriction_{Q_n(G,\phi)} \in$ Pol for every $n \in \mathbb{N}$. Since $\mathfrak{Q}(G, \phi)$ is an increasing union of the subgroups $Q_n(G, \phi)$, $\phi \restriction_{\mathfrak{Q}(G,\phi)} \in$ Pol by Proposition 5.3(d). □

Corollary 5.11. *For every algebraic flow (G, ϕ), we have $\mathfrak{Q}(G, \phi) \subseteq \mathrm{Pol}(G, \phi)$.*

Corollary 5.12. *Every locally quasi-periodic endomorphism has polynomial growth.*

6. Two characterizations of the Pinsker subgroup

Lemma 6.1. *Let (G, ϕ) be an algebraic flow.*

(a) *If $\phi \in \mathrm{Pol}_F$ for some $F \in [G]^{<\omega}$, then $H(\phi, F) = 0$.*
(b) *If $\phi \in$ Pol, then $h(\phi) = 0$.*

Proof. (a) By definition there exists $P_F(X) \in \mathbb{Z}[X]$ such that $\tau_{\phi,F}(n) \leq P_F(n)$ for every $n \in \mathbb{N}_+$. Then

$$H(\phi, F) = \lim_{n\to\infty} \frac{\log \tau_{\phi,F}(n)}{n} \leq \lim_{n\to\infty} \frac{\log P_F(n)}{n} = 0.$$

(b) Follows from (a). □

Corollary 6.2. *Let (G, ϕ) be an algebraic flow. Then $\mathfrak{Q}(G, \phi) \subseteq \mathrm{Pol}(G, \phi) \subseteq \mathbf{P}(G, \phi)$.*

Proof. The inclusion $\mathfrak{Q}(G, \phi) \subseteq \mathrm{Pol}(G, \phi)$ is proved in Corollary 5.11. By Lemma 6.1(b), $\mathrm{Pol}(G, \phi) \subseteq \mathbf{P}(G, \phi)$. □

The inclusion $\mathfrak{Q}(G, \phi) \subseteq \mathbf{P}(G, \phi)$ from Corollary 6.2 gives

Corollary 6.3. *Let (G, ϕ) be an algebraic flow. If ϕ has completely positive algebraic entropy, then ϕ is algebraically ergodic.*

We shall see below that this implication can be inverted (see Corollary 6.8).

Corollary 6.4. *Let (G, ϕ) be an algebraic flow. If G is torsion, then $\mathbf{P}(G, \phi) = t_\phi(G) = \mathrm{Pol}(G, \phi) = \mathfrak{Q}(G, \phi) = Q_1(G, \phi)$.*

Proof. By Proposition 3.7, $\mathbf{P}(G, \phi) = t_\phi(G) = Q_1(G, \phi) \subseteq \mathfrak{Q}(G, \phi)$, so Corollary 6.2 applies. □

The following theorem due to Kronecker is needed in the next proof:

Theorem 6.5 (*Kronecker Theorem [14]*)**.** *Let α be a non-zero algebraic integer and let $P(X) \in \mathbb{Z}[X]$ be its minimal polynomial over $\mathbb{Q}$. If all the roots of $P(X)$ have absolute value ≤ 1, then α is a root of unity.*

In the next theorem we show that for a non-trivial abelian group an endomorphism of zero algebraic entropy is not algebraically ergodic.

Theorem 6.6. *Let (G, ϕ) be an algebraic flow. If $G \neq 0$ and $h(\phi) = 0$, then $Q_1(G, \phi) \neq 0$.*

Proof. We split the proof in several steps, restricting the problem to the case of an automorphism of $\mathbb{Q}^n$ for some $n \in \mathbb{N}_+$.

(a) We can suppose that ϕ is injective. Indeed, if ϕ is not injective, then there are certainly non-zero quasi-periodic elements, as $\ker \phi \neq 0$.

(b) We can suppose that G is torsion-free. In fact, if G is torsion, $G = \mathbf{P}(G, \phi) = Q_1(G, \phi)$ by Proposition 3.7, and so $Q_1(G, \phi) \neq 0$. If G has non-trivial torsion elements, then $t(G) \neq 0$, and so $Q_1(t(G), \phi\restriction_{t(G)}) \neq 0$ by the torsion case.

(c) We can suppose that G is a divisible torsion-free abelian group. Indeed, by (b) we can assume that G is torsion-free. Let D be the divisible hull of G and $\widetilde{\phi} : D \to D$ the (unique) extension of ϕ to D. By Lemma 2.7, $h(\widetilde{\phi}) = 0$. Assume that $Q_1(D, \widetilde{\phi}) \neq 0$. Since G is essential in D and $Q_1(G, \phi) = Q_1(D, \widetilde{\phi}) \cap G$ by Lemma 4.3(a), it follows that also $Q_1(G, \phi) \neq 0$.

(d) We can suppose that G is a divisible torsion-free abelian group of finite rank. Indeed, if there exists a non-zero element $x \in G$ with $V(\phi, x)$ of infinite rank, then $h(\phi\restriction_{V(\phi,x)}) = \infty$, since $\phi\restriction_{V(\phi,x)}$ is conjugated to the right Bernoulli shift $\beta_{\mathbb{Z}}$ with $h(\beta_{\mathbb{Z}}) = \infty$ (see Example 2.6) and so Fact 2.5(b) applies. By Fact 2.5(a) this implies $h(\phi) = \infty$, against our hypothesis. Then $V(\phi, x)$ has finite rank for every $x \in G$. Moreover, each $V(\phi, x)$ is ϕ-invariant. So we can assume without loss of generality that G has finite rank and $G = V(\phi, x)$ for some $x \in G$.

(e) Suppose that G is a divisible torsion-free abelian group of finite rank $n > 0$. Then $G \cong \mathbb{Q}^n$ and by (a) we can assume that ϕ is an automorphism of G. Let $A = (a_{ij})$ be the $n \times n$ matrix over $\mathbb{Q}$ of ϕ, let $P(X)$ be the characteristic polynomial of A and λ_i the eigenvalues of A. In other words, $\phi = \sum_{i,j=1}^n a_{ij}\varepsilon_{ij}$, where $\varepsilon_{ij} \in \mathrm{End}(\mathbb{Q}^n)$ is defined (for $k, i, j = 1, 2, \ldots, n$) by $\varepsilon_{ij}(e_k) = \delta_{kj}e_i$, $(e_1, e_2, \ldots, e_n)$ is the canonical base of $\mathbb{Q}^n$ and δ_{kj} is Kronecker's symbol. Alternatively, if $\nu_i : \mathbb{Q} \to \mathbb{Q}^n$ and $p_j : \mathbb{Q}^n \to \mathbb{Q}$ are defined by $\nu_i(r) = re_i$ and $p_j(r_1, \ldots, r_n) = r_j$ for $r, r_1, \ldots, r_n \in \mathbb{Q}$, then $\varepsilon_{ij} = \nu_i \circ p_j$.

Let $\widehat{\phi} : \widehat{\mathbb{Q}}^n \to \widehat{\mathbb{Q}}^n$ be the adjoint automorphism of ϕ, where $\widehat{\mathbb{Q}}$ is the Pontryagin dual of $\mathbb{Q}$ (here we use the fact that $\widehat{\mathbb{Q}^n} \cong \widehat{\mathbb{Q}}^n$ and the isomorphism is natural, hence we can replace $\widehat{\mathbb{Q}^n}$ by $\widehat{\mathbb{Q}}^n$). Since $h(\phi)$ coincides with the topological entropy $h_{top}(\widehat{\phi})$ of $\widehat{\phi}$ by a theorem of Peters [16], our hypothesis $h(\phi) = 0$ implies that $h_{top}(\widehat{\phi}) = 0$ as well. The groups $\widehat{\mathbb{Q}}$ and $\widehat{\mathbb{Q}}^n$ are torsion-free and divisible [9, Corollary 3.3.8, Proposition 3.3.15], so they are $\mathbb{Q}$-vector spaces. Let us see that $\widehat{\phi}$ is described by the transposed matrix $A^t \in \mathrm{GL}_n(\mathbb{Q})$ of A, i.e., $\widehat{\phi}(x) = A^t x^t$ for $x = (x_1, \ldots, x_n) \in \widehat{\mathbb{Q}}^n$. Indeed, using the fact that the correspondence $\phi \mapsto \widehat{\phi}$ is $\mathbb{Q}$-linear (see [8] for the general case of locally compact modules over a commutative ring), from the equality $\phi = \sum_{i,j=1}^n a_{ij}\varepsilon_{ij}$ we deduce

$$\widehat{\phi} = \sum_{i,j=1}^n a_{ij}\widehat{\varepsilon}_{ij} = \sum_{i,j=1}^n a_{ij}\widehat{\nu_i \circ p_j} = \sum_{i,j=1}^n a_{ij}\widehat{p_j} \circ \widehat{\nu_i}. \tag{6.1}$$

Furthermore, one can easily check that $\widehat{\nu_i} : \widehat{\mathbb{Q}}^n \to \widehat{\mathbb{Q}}$ coincides with the ith projection and $\widehat{p_j} : \widehat{\mathbb{Q}} \to \widehat{\mathbb{Q}}^n$ coincides with the canonical embedding into the jth coordinate (so that $\widehat{p_j} \circ \widehat{\nu_i}$ is the endomorphism of $\widehat{\mathbb{Q}}^n$ that identically sends the ith copy of $\widehat{\mathbb{Q}}$ to the jth copy of $\widehat{\mathbb{Q}}$ and $\widehat{p_j} \circ \widehat{\nu_i}$ is trivial elsewhere). So (6.1) yields $\widehat{\phi}(x) = A^t x^t$ for $x = (x_1, \ldots, x_n) \in \widehat{\mathbb{Q}}^n$.

The eigenvalues and the characteristic polynomials of A and A^t coincide. According to the Yuzvinski Formula for the topological entropy of automorphisms of $\widehat{\mathbb{Q}}^n$ (see [15,22,25]), we have

$$h_{top}(\widehat{\phi}) = \log s + \sum_{|\lambda_i|>1} \log|\lambda_i|,$$

where s is the least common multiple of the denominators of the coefficients of $P(X)$. Then $h_{top}(\widehat{\phi}) = 0$ implies that $s = 1$ and that $|\lambda_i| \leq 1$ for every i. In other words, $P(X)$ is a monic polynomial with all roots of modulus ≤ 1. By Theorem 6.5, all the roots of $P(X)$ are roots of the unity. Then there exist $x \in G \setminus \{0\}$ and $m \in \mathbb{N}_+$ such that $\phi^m(x) = x$, that is, x is a non-zero periodic point of ϕ. In particular, $Q_1(G, \phi) \neq 0$. □

Remark 6.7 (*Added in April 2011*)**.** In item (e) of the proof of Theorem 6.6 we verify that for an automorphism $\phi : \mathbb{Q}^n \to \mathbb{Q}^n$, if $h(\phi) > 0$ then $Q_1(G, \phi) \neq 0$. Actually, we show a stronger result, that is,

$$h(\phi) = 0 \quad \text{implies that all the eigenvalues of } \phi \text{ are roots of unity.} \tag{6.2}$$

We deduce this result from two deep facts involving the topological entropy. One is the so-called Yuzvinski Formula for the topological entropy and the other is the Bridge Theorem proved by Peters. As a by-product we show in this way the so-called Algebraic Yuzvinski Formula, stating that $h(\phi) = \log s + \sum_{|\lambda_i|>1} \log |\lambda_i|$, where s is the least common multiple of the denominators of the coefficients of the characteristic polynomial $P(X)$ of ϕ and the λ_i are the eigenvalues of ϕ.

Since we are concerned with the simpler case of zero algebraic entropy, it would be desirable to avoid the use of these heavy theorems. This becomes possible now because of very recent results from [7] that produce exactly the required result with a direct algebraic proof. More precisely, (6.2) is a direct consequence of [7, Corollary 1.3] and [7, Corollary 1.4] covers item (e) of the proof of Theorem 6.6.

A direct proof of the Algebraic Yuzvinski Formula is now given in [12].

Theorem 6.6 allows us to characterize the algebraically ergodic algebraic flows.

Corollary 6.8. *Let (G, ϕ) be an algebraic flow. Then ϕ is algebraically ergodic if and only if ϕ has completely positive algebraic entropy.*

Proof. According to Corollary 6.3 it suffices to prove (equivalently) that

$$\mathfrak{Q}(G, \phi) = 0 \implies \mathbf{P}(G, \phi) = 0. \tag{6.3}$$

To this end, let $H = \mathbf{P}(G, \phi)$. Then $\mathfrak{Q}(H, \phi\restriction_H) = \mathfrak{Q}(G, \phi) \cap H$ by Lemma 4.3(b). By hypothesis $\mathfrak{Q}(G, \phi) = 0$, and so $\mathfrak{Q}(H, \phi\restriction_H) = 0$ as well. In particular, $Q_1(H, \phi\restriction_H) = 0$. So the assumption $\mathbf{P}(G, \phi) \neq 0$, along with Theorem 6.6, would give $h(\phi\restriction_H) > 0$, a contradiction. Hence, $\mathbf{P}(G, \phi) = 0$, and (6.3) is proved. □

Now we are in the position to prove our Main Theorem:

Theorem 6.9. *Let (G, ϕ) be an algebraic flow. Then $\mathfrak{Q}(G, \phi) = \mathrm{Pol}(G, \phi) = \mathbf{P}(G, \phi)$.*

Proof. By Corollary 6.2, $\mathfrak{Q}(G, \phi) \subseteq \mathrm{Pol}(G, \phi) \subseteq \mathbf{P}(G, \phi)$. To prove that $\mathbf{P}(G, \phi) \subseteq \mathfrak{Q}(G, \phi)$, let $\overline{\phi} : G/\mathfrak{Q}(G, \phi) \to G/\mathfrak{Q}(G, \phi)$ be the endomorphism induced by ϕ. By Proposition 4.11(a) $\mathfrak{Q}(G/\mathfrak{Q}(G, \phi), \overline{\phi}) = 0$ and so (6.3) gives

$$\mathbf{P}(G/\mathfrak{Q}(G, \phi), \overline{\phi}) = 0. \tag{6.4}$$

Let $\pi : G \to G/\mathfrak{Q}(G, \phi)$ be the canonical projection. Since $\pi(\mathbf{P}(G, \phi)) \subseteq \mathbf{P}(G/\mathfrak{Q}(G, \phi), \overline{\phi})$ by Lemma 3.3(b), from (6.4) we conclude that $\mathbf{P}(G, \phi) \subseteq \ker \pi = \mathfrak{Q}(G, \phi)$. □

The following is a direct consequence of Proposition 4.11(a) and Theorem 6.9.

Corollary 6.10. *Let (G, ϕ) be an algebraic flow. Then the induced endomorphism $\overline{\phi} : G/\mathbf{P}(G, \phi) \to G/\mathbf{P}(G, \phi)$ has $h(\overline{\phi}) \gg 0$, i.e., $\mathbf{P}(G/\mathbf{P}(G, \phi), \overline{\phi}) = 0$.*

Now we see that the endomorphisms with polynomial growth are precisely those of zero algebraic entropy.

Corollary 6.11. *Let (G, ϕ) be an algebraic flow. Then $h(\phi) = 0$ if and only if $\phi \in \mathrm{Pol}$. Consequently, $h(\phi) > 0$ if and only if there exists $F \in [G]^{<\omega}$ such that $\phi \notin \mathrm{Pol}_F$.*

Proof. Since $h(\phi) = 0$ if and only if $G = \mathbf{P}(G, \phi)$, and $\mathbf{P}(G, \phi) = \mathrm{Pol}(G, \phi)$ by Theorem 6.9, we can conclude that $h(\phi) = 0$ precisely when $\phi \in \mathrm{Pol}$. □

Corollary 6.11 can be stated in the following form that enhances the "local" ingredient F:

$$H(\phi, F) = 0 \quad \text{for every } F \in [G]^{<\omega} \text{ if and only if } \phi \in \mathrm{Pol}_F \text{ for every } F \in [G]^{<\omega}. \tag{6.5}$$

Indeed, $h(\phi) = 0$ is equivalent to $H(\phi, F) = 0$ for every $F \in [G]^{<\omega}$ and $\phi \in \mathrm{Pol}$ is equivalent to $\phi \in \mathrm{Pol}_F$ for every $F \in [G]^{<\omega}$. It is natural to ask if it possible to strengthen (6.5) by removing the universal quantifier. Now we prove this more precise result, providing an important dichotomy for the algebraic entropy with respect to a non-empty finite subset and for the growth of the cardinality of the trajectories of a non-empty finite subset.

Theorem 6.12. *Let (G, ϕ) be an algebraic flow and $F \in [G]^{<\omega}$. Then*

$$H(\phi, F) \begin{cases} > 0 & \textit{if and only if } \phi \in \mathrm{Exp}_F, \\ = 0 & \textit{if and only if } \phi \in \mathrm{Pol}_F. \end{cases}$$

Proof. We prove first that $H(\phi, F) > 0$ if and only if $\phi \in \mathrm{Exp}_F$. Note that both $H(\phi, F) > 0$ and $\phi \in \mathrm{Exp}_F$ imply $|F| \geq 2$; indeed, if $|F| = 1$, then $\tau_{\phi,F}(n) = 1$ for every $n \in \mathbb{N}_+$.

Assume that $H(\phi, F) = a > 0$. Consequently, there exists $m \in \mathbb{N}_+$ such that $\log \tau_{\phi,F}(n) > n \cdot \frac{a}{2}$ for every $n > m$. Then $\tau_{\phi,F}(n) > e^{n\cdot\frac{a}{2}}$ for every $n > m$. Since $|F| \geq 2$, $\tau_{\phi,F}(n) \geq 2$ for every $n \in \mathbb{N}_+$; in particular, $\tau_{\phi,F}(n) \geq (\sqrt[m]{2})^n$ for every $n \leq m$. For $b = \min\{\sqrt[m]{2}, e^{\frac{a}{2}}\}$, we have $\tau_{\phi,F}(n) \geq b^n$ for every $n \in \mathbb{N}_+$, and this proves that $\phi \in \mathrm{Exp}_F$.

Suppose now that $\phi \in \mathrm{Exp}_F$. Then there exists $b \in \mathbb{R}_+$, $b > 1$, such that $\tau_{\phi,F}(n) \geq b^n$ for every $n \in \mathbb{N}_+$. Hence $H(\phi, F) \geq \log b > 0$.

By Lemma 6.1(a) if $\phi \in \mathrm{Pol}_F$, then $H(\phi, F) = 0$. If $H(\phi, F) = 0$, by Lemma 2.8 $h(\phi\restriction_{V(\phi,F)}) = 0$. Corollary 6.11 implies in particular that $\phi \in \mathrm{Pol}_F$. □

Note that this more precise form of (6.5) follows from (6.5) (since in the proof of Theorem 6.12 we apply Corollary 6.11); so they are equivalent.

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